

ON THE K PROPERTY FOR MAHARAM EXTENSIONS OF BERNOULLI SHIFTS AND A QUESTION OF KRENGEL

ABSTRACT. We show that the Maharam extension of a type III, conservative and non singular K Bernoulli is a K -transformation. This together with the fact that the Maharam extension of a conservative transformation is conservative gives a negative answer to Krengel's and Weiss's questions about existence of a type II_∞ or type III_λ with $\lambda \neq 1$ Bernoulli shift. A conservative non singular K , in the sense of Silva and Thieullen, Bernoulli shift is either of type II_1 or of type III_1 .

1. INTRODUCTION

Let T be an invertible non singular transformation of the probability space (X, \mathcal{B}, μ) . The Maharam extension \tilde{T} of T is a measure preserving transformation which is a skew product extension of T with the Radon Nykodym cocycle. It is well known that the Maharam extension is ergodic if and only if T is of Krieger type III_1 , see below. Here we show that in the case when T is a conservative non singular Bernoulli shift which satisfies the K -property as in [ST] then the Maharam extension is a K -transformation. Thus the Maharam extension is weak mixing in the sense that $T \times S$ is ergodic for every ergodic probability preserving transformation S and it has a countable Lebesgue spectrum.

This type of non singular Bernoulli shifts was considered first in [Kre] where a shift without an absolutely continuous invariant probability was constructed. Later Hamachi in [Ham] constructed an ergodic shift without an absolutely continuous σ -finite invariant measure a.k.a type III . Krengel [Kre] asked the question whether there exists a shift with an absolutely continuous invariant σ -finite measure but no such probability (a.k.a type II_∞). The type III transformation can be further classified into orbit equivalence classes according to their ratio set. In [Kos] a Bernoulli shift which is of Krieger type III_1 was constructed. In a presentation of that result Benjy Weiss asked whether there are type III shifts of different Krieger types. As a corollary of the K property of the Maharam extension we get a dichotomy. Namely an ergodic non singular K Bernoulli shift is either of type II_1 when the measure is equivalent to a stationary product measure or of type III_1 .

The proof makes use of the fact that since the Radon Nykodym cocycle is measurable with respect to the σ -algebra $\mathcal{B}_{\{0,1\}^{\mathbb{N}}}$, the Maharam extension is the natural extension of a skew product σ_φ of the one sided shift. Thus it is enough to show that the tail equivalence relation of the non invertible skew product is ergodic. This is done by showing that the tail equivalence relation of σ_φ is the orbit

This research was supported by THE ISRAEL SCIENCE FOUNDATION grant No. 1114/08.

equivalence relation of the Maharam extension of the odometer and proving that the odometer with the one sided measure is of type III₁.

One step in the proof that the corresponding odometer action is type III₁ is to show that for shift conservative product measures we have two subsequences $n_k \rightarrow \infty$ and $m_k \rightarrow -\infty$ for which

$$\lim_{k \rightarrow \infty} P_{n_k} = \lim_{k \rightarrow \infty} P_{m_k}.$$

The question arises whether for conservative shifts the limit needs to exist? We give an example of a conservative shift with

$$\liminf_{k \rightarrow \infty} P_k(0) < \limsup_{k \rightarrow \infty} P_k(0),$$

thus answering this question on the negative.

Acknowledgments. I would like to thank my advisor Prof. Jon Aaronson for many valuable suggestions. I would also like to thank Prof. Ulrich Krengel for sending me a copy of Michael Grewe's master thesis.

2. PRELIMINARIES

2.1. Non Singular Ergodic Theory. Let (X, \mathcal{B}, μ) be a standard measure space. In what follows all equalities of sets are modulo the measure on the space.

A measurable transformation $T : X \rightarrow X$ is non singular if μ and $T_*\mu = \mu \circ T^{-1}$ are equivalent, meaning that they have the same collection of null sets. In the case when T is invertible there exists the Radon Nykodym derivatives

$$T^{n'}(x) := \frac{d\mu \circ T^n}{d\mu}(x).$$

When $T_*\mu = \mu$ we say that T is μ preserving or μ is T invariant. A transformation is *ergodic* if $T^{-1}A = A$ implies $A \in \{\emptyset, X\}$. A set $A \in \mathcal{B}$ is *wandering* if $\{T^{-n}A\}_{n=1}^{\infty}$ are disjoint. Denote by \mathfrak{D} the (measurable) union of all wandering sets, it's complement is denoted by \mathfrak{C} and is called the conservative part. In the case where $\mathfrak{D} = X$ we say that T is *dissipative*. If $\mathfrak{C} = X$ we say that T is *conservative*. By Hopf's theorem [Aa, Prop. 1.3.1.]

$$(2.1) \quad \begin{aligned} \mathfrak{D} &= \left\{ x \in X : \sum_{n=1}^{\infty} T^{n'}(x) < \infty \right\} \\ \mathfrak{C} &= \left\{ x \in X : \sum_{n=1}^{\infty} T^{n'}(x) = \infty \right\}. \end{aligned}$$

An invertible transformation T satisfies the *K-property* if there exists a sub- σ algebra $\mathcal{F} \subset \mathcal{B}$ such that $T^{-1}\mathcal{F} \subset \mathcal{F}$, $\bigcap_{n \in \mathbb{Z}} T^n \mathcal{F} = \{\emptyset, X\}$ and $\bigvee_{n=1}^{\infty} T^{-n} \mathcal{F} = \mathcal{B}$. If T is measure preserving and K then T

is either conservative or totally dissipative. This property remains true in the case of non singular Bernoulli shifts, see Lemma 5.1 or [Gre].

A measure preserving transformation $(Y, \mathcal{B}_Y, \nu, S)$ is an extension of $(X, \mathcal{B}_X, \mu, T)$ (equivalently T is a factor of X) if there exists a measurable map $\pi : Y \rightarrow X$ such that $\pi^{-1}\mathcal{B}_X \subset \mathcal{B}_Y$, $\pi \circ S = T \circ \pi$ and $\pi_*\nu = \mu$. Given a non-invertible measure preserving transformation $(X, \mathcal{B}_X, \mu, T)$, the *natural extension* of T is an invertible measure preserving transformation \tilde{T} which is minimal in the sense that

$$\bigvee_{n=1}^{\infty} \tilde{T}^n \pi^{-1} \mathcal{B}_X = \mathcal{B}_{\tilde{X}},$$

where $\pi : \tilde{X} \rightarrow X$ is the factor map.

2.2. Cocycles and skew product extensions. A function $\varphi : \mathbb{N} \times X \rightarrow \mathbb{R}$ (or $\mathbb{Z} \times X \rightarrow \mathbb{R}$ when T is invertible) is a *cocycle* if for every $n, m \in \mathbb{N}$ and almost every $x \in X$,

$$(2.2) \quad \varphi_{n+m}(x) = \varphi_n(x) + \varphi_m(T^n x).$$

Given a function $\varphi : X \rightarrow \mathbb{R}$ we can define the cocycle

$$\forall n \in \mathbb{N}, \varphi_n(x) = \varphi(x) + \varphi \circ T(x) + \dots + \varphi \circ T^{n-1}(x),$$

and the *skew product extension* $T_\varphi : (X \times \mathbb{R}, \mathcal{B}_X \otimes \mathcal{B}_{\mathbb{R}}, \mu \times e^s ds)$ of T with φ by

$$T_\varphi(x, y) := (Tx, y + \varphi(x)).$$

Definition. The set of *essential values* for φ is

$$e(T, \varphi) = \{t \in \mathbb{R} : \forall \epsilon > 0, \forall A \in (\mathcal{B}_X)_+, \exists n \in \mathbb{N} \text{ s.t. } \mu(A \cap T^{-n}A \cap [|\varphi_n - a| < \epsilon]) > 0\}$$

It follows from the cocycle equation (2.2) that the set of essential values is a closed subset (under addition) of \mathbb{R} and therefore it is of the form $\emptyset, \{0\}, \{0\} \cup a\mathbb{Z}$ ($a \in \mathbb{R}$) or \mathbb{R} . The skew product T_φ is ergodic if and only if T is ergodic and $e(S, \varphi) = \mathbb{R}$.

We will be interested in the *Maharam extension* \tilde{T} which is the skew product extension of an invertible transformation $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ with $\varphi(x) = \log T'(x)$, the Radon-Nykodym cocycle. In the case when the Maharam extension is ergodic we say that T is of type III₁ ($e\left(T, \log \frac{d\mu \circ S}{d\mu}\right) = \mathbb{R}$ and T is ergodic). In the case where T is conservative and there exists a μ -equivalent σ -finite invariant measure the essential value set is $e\left(T, \log \frac{d\mu \circ S}{d\mu}\right) = \{0\}$.

2.3. The tail and the orbital equivalence relation of a transformation. For a more detailed discussion of the contents of this subsection see [KM].

Let (X, \mathcal{B}_X) be a standard measure space. An equivalence relation on X is a set $\mathcal{R} \subset X \times X$ such that the relation $x \sim y$ if and only if $(x, y) \in \mathcal{R}$ is an equivalence relation. It is measurable if

$\mathcal{R} \subset \mathcal{B}_X \otimes \mathcal{B}_X$. Given an equivalence relation \mathcal{R} and a set $A \in \mathcal{B}_X$, the *saturation* of A is the set

$$\mathcal{R}(A) := \bigcup_{x \in A} R_x,$$

where $\mathcal{R}_x := \{y \in X : (x, y) \in R\}$. Given a measure μ on X , we say that \mathcal{R} is μ -ergodic if for each $A \in \mathcal{B}_X$,

$$\mathcal{R}(A) \in \{\emptyset, X\} \text{ mod } \mu.$$

An equivalence relation is finite (respectively countable) if for all $x \in X$, \mathcal{R}_x is a finite (countable) set. It is hyperfinite if there exists an increasing sequence of finite subequivalence relation $E_1 \subset E_2 \subset \dots \subset \mathcal{R}$ such that

$$\mathcal{R} = \bigcup_{n=1}^{\infty} E_n.$$

Given a non singular non-invertible transformation $(X, \mathcal{B}_X, \nu, S)$ we define the *orbit equivalence relation* on $X \times X$

$$\mathcal{R}_S := \{(y_1, y_2) \in X \times X : \exists n, m \in \mathbb{N}, S^n y_1 = S^m y_2\}.$$

and the *tail relation*, which we denote by $\mathcal{T}(S)$, by

$$\mathcal{T}(S) = \{(y_1, y_2) : \exists n \in \mathbb{N}, S^n y_1 = S^n y_2\}.$$

A transformation is *exact* if for all $A \in \mathcal{B}$,

$$\mathcal{T}(S)_A \in \{\emptyset, Y\} \text{ mod } \nu$$

By [We, Sls] an equivalence relation is hyperfinite if and only if it is an orbit relation of a non singular transformation. Since $\mathcal{T}_S \subset \mathcal{R}_S$ and \mathcal{R}_S is hyperfinite we have that \mathcal{T}_S is hyperfinite. Therefore there exists a non-singular transformation V of (Y, \mathcal{B}_Y, ν) , which we call the *tail action of S* , such that

$$\mathcal{R}_V = \mathcal{T}_S.$$

It follows that S is exact if and only if V is ergodic.

A function $\hat{\varphi} : \mathcal{R} \rightarrow \mathbb{R}$ is an *orbital cocycle* if for every $x, y, z \in X$ in the same equivalence class of \mathcal{R} ,

$$\hat{\varphi}(x, y) = \hat{\varphi}(x, z) + \hat{\varphi}(z, y).$$

To every function $\varphi : X \rightarrow \mathbb{R}$ corresponds an orbital cocycle $\hat{\varphi}$ on \mathcal{T}_S (notice that the sum is actually a finite sum) defined by

$$\hat{\varphi}(y_1, y_2) := \sum_{n=0}^{\infty} \{\varphi(S^n y_1) - \varphi(S^n y_2)\}, \quad (y_1, y_2) \in \mathcal{T}(S).$$

and the \mathcal{R}_V -cocycle ψ defined by

$$\psi(y) = \hat{\varphi}(y, Vy).$$

The following fact shows that the skew product S_φ is exact if and only if V_ψ is ergodic where V is the tail action of S and ψ is its corresponding cocycle.

Fact 2.1. [ANS] *Let $(Y, \mathcal{B}_Y, \nu, S)$ be a non singular and non-invertible transformation and $(Y, \mathcal{B}_Y, \nu, V)$ its associated tail action. Let $\varphi : Y \rightarrow \mathbb{R}$ be a function and ψ the corresponding \mathcal{R}_V cocycle. Then*

$$\mathcal{T}_{S_\varphi} = \mathcal{R}_{V_\psi}.$$

2.4. The Zero Type property and dissipative transformations: Given two measures on (X, \mathcal{B}) we can define the Hellinger Integral [Kak, Kos] by

$$\rho(\mu, \nu) = \int_X \sqrt{\frac{d\mu}{d\lambda}} \sqrt{\frac{d\nu}{d\lambda}} d\lambda$$

where λ is any measure on X such that $\nu \ll \lambda$ and $\mu \ll \lambda$.

If T is a non singular transformation of (X, \mathcal{B}, μ) then since $T_*^n \mu \sim \mu$ we have

$$\rho(n) := \rho(\mu, T_*^n \mu) = \int_X \sqrt{T^{n'}(x)} d\mu(x).$$

A transformation is Zero-Type (a.k.a. mixing) if the maximal spectral type of its Koopman operator defined by

$$\forall f \in L^2(X, \mu), U_T f := \sqrt{T'} \cdot f \circ T$$

is a Rajchman measure. This is equivalent to the condition: For every $f, g \in L^2(X, \mu)$,

$$\int_X U_T^n f \cdot \bar{g} d\mu \xrightarrow{n \rightarrow \infty} 0.$$

Note that when T is probability preserving one needs to restrict the class of functions to $L^2(X, \mu) \ominus \mathbb{C}$.

The following proposition is standard.

Proposition 2.2. *Given a non singular transformation (X, \mathcal{B}, μ, T) , the following are equivalent:*

- (1) T is zero-type.
- (2) $\lim_{n \rightarrow \infty} \rho(\mu, T_*^n \mu) = 0$.
- (3) $T^{n'}$ tends to 0 in μ measure. That is for every $\epsilon > 0$,

$$\mu \left(x : \left| T^{n'}(x) \right| > \epsilon \right) \xrightarrow{n \rightarrow \infty} 0.$$

The next lemma will be used to get a necessary criterion for conservativity of Bernoulli shifts. .

Lemma 2.3. *If (X, \mathcal{B}, μ, T) is zero type and $\sum_{n=1}^{\infty} \rho(\mu, T_*^n \mu) < \infty$ then T is dissipative.*

Proof. Since

$$\mu\left(\left|T^{n'}\right| > 1\right) \leq \int_X \sqrt{T^{n'}} d\mu = \rho(\mu, T_*^n \mu)$$

and the right hand side is summable, it follows from the Borel Cantelli lemma that for almost every $x \in X$ there exists $N(x) \in \mathbb{N}$ such that for every $n > N(x)$,

$$T^{n'}(x) \leq \sqrt{T^{n'}(x)} \leq 1.$$

In addition the summability condition on $\rho(\mu, T_*^n \mu)$ ensures that

$$\sum_{n=1}^{\infty} \sqrt{T^{n'}} < \infty \text{ a.e. } d\mu.$$

Therefore by comparison of sums we have that

$$\sum_{n=1}^{\infty} T^{n'}(x) < \infty \text{ a.e. } d\mu$$

and so T is dissipative. □

3. HALF STATIONARY BERNOULLI SHIFTS

3.1. Non Singular Bernoulli Shift. Let $X = \{0, 1\}^{\mathbb{Z}}$, $\mathcal{B} = \mathcal{B}_X$, $X^+ = \{0, 1\}^{\mathbb{N}}$ and $\mathcal{B}^+ = \mathcal{B}_{X^+}$. We will write σ for the one-sided shift on X^+ and T for the full shift on X .

A product measure $P = \prod_{k=-\infty}^{\infty} P_k \in \mathcal{P}(X)$ is **half stationary** if there exists $p \in (0, 1)$ such that for all $k \leq 0$,

$$P_k(0) = 1 - P_k(1) = p.$$

We will consider the case $p = \frac{1}{2}$. The case of general p being similar.

Thus the general form of a half stationary product measure (with $p = \frac{1}{2}$) is

$$(3.1) \quad P_k(0) = 1 - P_k(1) = \begin{cases} \frac{1-a_i}{2} & k \in \mathbb{N} \\ \frac{1}{2} & k \leq 0 \end{cases},$$

where $a_i \in (-1, 1)$.

Let $P^+ = \prod_{k=1}^{\infty} P_k$ denote the measure of P restricted to X^+ . If P is half stationary, then the full shift T is the natural extension, in the sense of Silva and Thieullen [ST], of the one sided shift $\left(X^+, \mathcal{B}, P^+ = \prod_{k=1}^{\infty} P_k, \sigma\right)$. Since by Kolmogorov's 0-1 Law the one sided shift is exact, the full shift

is a K -transformation. Conversely every K -Bernoulli shift such that T' is \mathcal{B}^+ measurable is a shift with a half stationary measure. We call such transformations non-singular K -shifts.

The following gives conditions on the product measures so that the shift is non singular and ergodic.

Theorem 3.1. *Let P be of the form (3.1). Then*

(1) *The shift (X, \mathcal{B}, P, T) is non singular if and only if for all $n \in \mathbb{N}$, $|a_n| \neq 1$ and*

$$(3.2) \quad \sum_{k=0}^{\infty} \left\{ \left(\sqrt{P_k(0)} - \sqrt{P_{k+1}(0)} \right)^2 + \left(\sqrt{P_k(1)} - \sqrt{P_{k+1}(1)} \right)^2 \right\} < \infty.$$

(2) *For every $n \in \mathbb{N}$,*

$$T^{n'}(x) = \frac{dP \circ T^n}{dP}(x) = \prod_{k=1}^{\infty} \frac{P_{k-n}(w_k)}{P_k(w_k)}.$$

(3) *If the shift is conservative then it is ergodic.*

(4) *There is an absolutely continuous invariant probability if and only*

$$\sum_{k=1}^{\infty} a_k^2 < \infty.$$

(5) *There exists constants $c, C > 0$ such that*

$$(3.3) \quad c \cdot d(P, T_*^n P) \leq -\log(\rho(P, T_*^n P)) \leq C \cdot d(P, T_*^n P)$$

where

$$d\left(\prod P_i, \prod Q_i\right) = \sum_{i \in \mathbb{Z}} \left\{ \left(\sqrt{P_i(0)} - \sqrt{Q_i(0)} \right)^2 + \left(\sqrt{P_i(1)} - \sqrt{Q_i(1)} \right)^2 \right\}.$$

Proof. (1) and (2) follow from Kakutani's Theorem, [Kak] on equivalence of product measures. Parts (3) and (4) are in [Kre]. (5) is an observation of Kakutani. \square

3.1.1. *The Odometer as the tail action of the shift.* We will also consider the odometer action τ on X^+ given by

$$\tau \left(\underbrace{1, 1, \dots, 1}_{n\text{-times}}, 0, w \right) = \left(\underbrace{0, 0, \dots, 0}_{n\text{-times}}, 1, w \right).$$

The odometer and the one sided shift satisfy

$$\mathcal{R}_\tau = \mathcal{T}_\sigma.$$

A calculation shows that

$$\tau'(x) = \frac{P_{\phi(x)}(1)}{P_{\phi(x)}(0)} \cdot \prod_{k=1}^{\phi(x)-1} \frac{P_k(0)}{P_k(1)}$$

where

$$\phi(x) := \min \{n \geq 1 : x_n = 0\}.$$

The odometer satisfies the so called *Odometer Property*, which states that for every $N \in \mathbb{N}$ and $x \in X^+$,

$$\left\{ \left(\left(\tau^k x \right)_1, \left(\tau^k x \right)_1, \dots, \left(\tau^k x \right)_n \right) : k = 0, 1, \dots, 2^N - 1 \right\} = \{0, 1\}^N.$$

Using this fact one shows that for every $n \in \mathbb{N}$,

$$(3.4) \quad \tau^{(2^n)'}(x) = \tau' \circ \sigma^n(x).$$

This can also be deduced from the fact that for all $n \in \mathbb{N}$ and $j \in \{1, \dots, n\}$,

$$(\tau^{2^n}(x))_j = x_j.$$

This property plays a crucial role in calculating the essential values of the odometer action. See the proof of Lemma 3.4 below.

3.2. Statement of the main theorem and the Answer to Krengel's question.

Theorem 3.2. For every (X, \mathcal{B}, P, T) a conservative and non singular K -shift without an absolutely continuous invariant measure the Maharam extension is a K -transformation.

As a corollary we get a negative answer to Krengel's question for non singular K -shifts.

Corollary 3.3. A conservative, ergodic, K -non singular Bernoulli shift is either of type III_1 or type II_1 .

Proof. Assume that there exists no a.c.i.p. By Maharam's theorem, the Maharam extension is conservative and by Theorem 3.2 it is K σ -finite measure preserving transformation. Therefore by [Par] it is ergodic and so the shift is of type III_1 . \square

3.3. The proof of Theorem 3.2. By Theorem 2, the Radon-Nykodym cocycle $\varphi(x) := \log T'(x)$ is \mathcal{B}^+ measurable.

It follows that the Maharam extension of T is the natural extension of the skew product $(X^+ \times \mathbb{R}, \mathcal{B}^+ \otimes \mathcal{B}_{\mathbb{R}}, P^+ \otimes e^s ds, \sigma_\varphi)$. Since a transformation is K if and only if it is a natural extension of an exact transformation, in order to show that the Maharam extension of the two sided shift is K , we will show that the skew product extension $\sigma_{\log T'}$ is exact.

This will be done in two steps. First we show that the odometer $(X^+, \mathcal{B}^+, P^+, \tau)$ is of type III_1 and then we show that

$$\mathcal{T}(\sigma_{\log T'}) = \mathcal{R}(\tau_{\log \tau'}),$$

thus the tail action is ergodic.

Lemma 3.4. *Let P be as in (3.1). If the shift is conservative and there exists no a.c.i.p then:*

- (1) *There exists a subsequence $\{a_{n_k}\}$ such that $\lim_{k \rightarrow \infty} a_{n_k} = 0$.*
- (2) *The odometer $(X^+, \mathcal{B}^+, P^+, \tau)$ is of type III₁.*

Proof. Denote by

$$\mathfrak{A} = \left\{ a \in \mathbb{R} : \exists n_k \rightarrow \infty, a_{n_k} \xrightarrow{k \rightarrow \infty} a \right\}$$

the set of limit points of the sequence $\{a_n\}$.

(1) Assume that $0 \notin \mathfrak{A}$. We will show that then $\sum_{n=1}^{\infty} \rho(P, T_*^n P) < \infty$ and so by Lemma 2.3 T is dissipative.

Since 0 is not a limit point of $\{a_n\}$, there exists an $\epsilon > 0$ and $N \in \mathbb{N}$ such that for all $i > N$,

$$(3.5) \quad \sqrt{\frac{1-a_i}{2}} - \sqrt{\frac{1}{2}} > \epsilon.$$

Therefore for every $n > N$,

$$\begin{aligned} d(P, T_*^n P) &= \sum_{i \in \mathbb{Z}} \left\{ \left(\sqrt{P_i(0)} - \sqrt{P_{i-n}(0)} \right)^2 + \left(\sqrt{P_i(1)} - \sqrt{P_{i-n}(1)} \right)^2 \right\} \\ &\geq \sum_{i=N}^n \left(\sqrt{P_i(0)} - \sqrt{P_{i-n}(0)} \right)^2 \\ &= \sum_{i=N}^n \left(\sqrt{\frac{1-a_i}{2}} - \sqrt{\frac{1}{2}} \right)^2. \end{aligned}$$

The last equality follows from the fact that $P_k = (\frac{1}{2}, \frac{1}{2})$ for $k \in \mathbb{Z} \setminus \mathbb{N}$. Therefore by (3.5) we have that

$$d(P, T_*^n P) \geq (n - N) \epsilon^2.$$

The conclusion follows from (3.3) and Lemma 2.3.

(2) Let P be a half stationary product measure such that the shift is conservative and there is no a.c.i.p.

One can show that we can choose a subsequence such that $\lim_{n \rightarrow \infty} a_{n_k} = 0$ and $\sum_{k=1}^{\infty} a_{n_k}^2 = \infty$ and then use standard techniques.

Alternatively we can argue as follows: Since there is no a.c.i.p. then

$$\sum_{n=1}^{\infty} a_n^2 = \infty.$$

Therefore if $\mathfrak{A} = \{0\}$ ($\lim a_n = 0$) then the odometer is of type III₁ by [DKQ, Prop. 3.1.].

Otherwise there is $0 < \alpha < 1$ such that $\{0, \alpha\} \subset \mathfrak{A}$. It follows from the non-singularity condition (3.2) that

$$[0, \alpha] \subset \mathfrak{A}.$$

We show that $e(\tau, \log \tau') = \mathbb{R}$ by showing that for every $p \in \mathfrak{A} \setminus \{-1, 1\}$,

$$\log \frac{1+p}{1-p} \in e(\tau, \log \tau'),$$

so the set of essential values contains an interval. This will be done by establishing the conditions of [DKQ, Lemma 2.1].

Let $p \in \mathfrak{A}$ and $a_{n_k} \xrightarrow{k \rightarrow \infty} p$.

Let

$$C = [c]_1^n := \{x \in X^+ : x_i = c_i \ \forall i \in [1, n]\}.$$

be a cylinder set and write

$$C_{n_k} = C \cap \{x \in X^+ : x_{n_k} = 0\}.$$

It follows from (3.4) that for every $k \in \mathbb{N}$ such that $n_k > n$,

$$\log \tau^{(2^{n_k})'} \Big|_{C_{n_k}} = \log \frac{1 + a_{n_k}}{1 - a_{n_k}}.$$

Therefore

$$\begin{aligned} (3.6) \quad P^+ \left(C \cap \tau^{-2^{n_k}} C \cap \left[\log \tau^{(2^{n_k})'} = \log \frac{1 + a_{n_k}}{1 - a_{n_k}} \right] \right) &\geq P^+(C_{n_k}) \\ &= \left(\frac{1 - a_{n_k}}{2} \right) P^+(C). \end{aligned}$$

Given $\epsilon > 0$, we can choose k large enough such that

$$\left(\frac{1 - a_{n_k}}{2} \right) > \frac{1 - p}{4} := \beta > 0.$$

and

$$\left| \log \frac{1 + a_{n_k}}{1 - a_{n_k}} - \log \frac{1 + p}{1 - p} \right| < \epsilon.$$

Then by (3.6) we get

$$P^+ \left(C \cap \tau^{-2^{n_k}} C \cap \left[\left| \log \tau^{(2^{n_k})'} - \log \frac{1 + p}{1 - p} \right| < \epsilon \right] \right) \geq \beta P^+(C).$$

Thus the conditions of [?, Lemma 2.1] are satisfied with

$$\gamma = \left(\underbrace{0, \dots, 0}_{n_k-1}, 1, \underline{0} \right)$$

and

$$\mathcal{U} = C_{n_k}.$$

Hence $\log \frac{1+p}{1-p}$ is an essential value for $\log \tau'$.

□

Lemma 3.5. *Let P be defined by (3.1), then*

$$\psi(x) = \log \tau'(x),$$

where $\psi(x)$ is the tail-cocycle corresponding to $\varphi = \log T'$.

Proof. Since

$$\sigma^n x = \sigma^n \tau x \iff n \geq \phi(x)$$

it follows that

$$\psi(x) = \sum_{k=0}^{\phi(x)-1} \left\{ \varphi(\sigma^k x) - \varphi(\sigma^k \tau x) \right\} = \varphi_{\phi(x)}(x) - \varphi_{\phi(x)}(\tau x).$$

This together with Theorem 3.1 and the fact that

$$(\tau x)_k = \begin{cases} 1 - x_k, & k \leq \phi(x) \\ x_k & k > \phi(x) \end{cases},$$

yields

$$\begin{aligned} \psi(x) &= \log \left(\prod_{k=1}^{\phi(x)} \left[\frac{P_{k-\phi(x)}(x_k)}{P_k(x_k)} \middle/ \frac{P_{k-\phi(x)}(1-x_k)}{P_k(1-x_k)} \right] \right) \\ &= \log \left(\prod_{k=1}^{\phi(x)} \left[\frac{P_{k-\phi(x)}(x_k)}{P_{k-\phi(x)}(1-x_k)} \cdot \frac{P_k(1-x_k)}{P_k(x_k)} \right] \right). \end{aligned}$$

Since for all $k < 0$, $P_k \equiv (1/2, 1/2)$,

$$\forall k \leq \phi(x), \frac{P_{k-\phi(x)}(x_k)}{P_{k-\phi(x)}(1-x_k)} = 1,$$

we see that

$$\psi(x) = \log \left(\prod_{k=1}^{\phi(x)} \frac{P_k(1-x_k)}{P_k(x_k)} \right) = \log \tau'(x).$$

□

Proof of Theorem 3.2. Since the Maharam extension \tilde{T} is the natural extension of σ_φ , we need to show that σ_φ is exact.

The odometer τ is the tail action of the shift σ . It follows from Lemma 3.5 that,

$$\mathcal{T}(\sigma_\varphi) = \mathcal{R}(\pi_{\log \tau'}).$$

By Lemma 3.4 $\pi_{\log \tau'}$ is ergodic (τ is type III₁) and therefore σ_φ is exact. □

4. EXAMPLES

In [Kre, Ham] examples of conservative shifts were constructed without an a.c.i.p. It follows from Theorem 3.2 that the Maharam extension is K and that those shifts are of type III₁. In these examples one has

$$(4.1) \quad \lim_{n \rightarrow \infty} P_n(0) = \frac{1}{2}.$$

We will give two more examples here. One of a dissipative half stationary shift with

$$\lim_{n \rightarrow \infty} P_n(0) = \frac{1}{2}$$

which shows that (4.1) is not sufficient for conservativity. The other is a conservative half stationary product measure with

$$\liminf_{n \rightarrow \infty} P_k(0) = \frac{1}{4}, \quad \limsup_{n \rightarrow \infty} P_k(0) = \frac{1}{2},$$

Together those examples show that Lemma 3.4.1 is all we can say about limit points of a_n .

Remark 4.1. Michael Grewe in his Master thesis [ref:] has constructed a different example of a dissipative shift with $P_k(0) \rightarrow \frac{1}{2}$. His method relies on the strong law of large numbers and an inductive construction. We include here a new example as the method of proof and the measure are more simple.

4.1. Dissipative example. Define a product measure by

$$P_n(0) = \begin{cases} \frac{1}{2} - \frac{2}{n}, & n \geq 2 \\ \frac{1}{2} & n < 2 \end{cases}.$$

Since

$$\sum_{k=0}^{\infty} \left\{ \left(\sqrt{P_k(0)} - \sqrt{P_{k+1}(0)} \right)^2 + \left(\sqrt{P_k(1)} - \sqrt{P_{k+1}(1)} \right)^2 \right\} < \infty,$$

the shift $(\{0, 1\}^{\mathbb{Z}}, P, T)$ is non singular. In addition

$$\begin{aligned} d(P, P \circ T^n) &\geq \sum_{k=0}^n \left\{ \left(\sqrt{P_k(0)} - \sqrt{P_{k-n}(0)} \right)^2 + \left(\sqrt{P_k(1)} - \sqrt{P_{k-n}(1)} \right)^2 \right\} \\ &= \sum_{k=2}^n \left\{ 2 - \sqrt{1 - \frac{4}{k}} - \sqrt{1 + \frac{4}{k}} \right\}. \end{aligned}$$

It follows from the Taylor expansion of $\sqrt{1+x}$ that

$$2 - \sqrt{1 - \frac{2}{k}} - \sqrt{1 + \frac{2}{k}} = \frac{2\sqrt{2} - 1}{k} + O_{k \rightarrow \infty} \left(\frac{1}{k^2} \right).$$

Therefore there exists a constant $C \in \mathbb{R}$ such that

$$d(P, P \circ T^n) \geq (2\sqrt{2} - 1) \sum_{k=2}^n \frac{1}{k} + C.$$

Since $\sum_{k=2}^n \frac{1}{k} \propto \log(n)$ and $\log \rho(P, P \circ T^n) \propto d(P, P \circ T^n)$, it follows that

$$\sum_{n=1}^{\infty} \rho(P, P \circ T^n) < \infty.$$

By Lemma 2.3 the shift is dissipative.

4.2. The “weird” conservative example. Given $k \in \mathbb{N}$ set

$$\lambda_n^{(k)} = \begin{cases} 1 + \frac{n}{2^k}, & n \in [0, 2^{k-1}] \\ 2 - \frac{n}{2^k}, & n \in [2^{k-1}, 2 \cdot 2^{k-1}] \\ 1, & \text{otherwise} \end{cases}.$$

and let $P^{(k)}$ be the product measure on X with factor measures

$$P_n^{(k)}(1) = \frac{\lambda_n^{(k)}}{1 + \lambda_n^{(k)}} = 1 - P_n^{(k)}(0).$$

Our example of a conservative product measure with

$$\limsup_{k \rightarrow \infty} P_k(1) = \frac{3}{4} > \frac{1}{2} = \liminf_{k \rightarrow \infty} P_k(1)$$

consists of large intervals where $P_k(0)$ is exactly $\frac{1}{2}$ followed by large intervals of the form $[N, N + 2^k]$ where $P_n(1)$ equals $P_{n-N}^{(k)}(1)$ (a slow increase to $\frac{3}{4}$ followed by a small decrease back to $\frac{1}{2}$). Then this segment is followed by a larger segment where $P_k(0) = \frac{1}{2}$ and so on. The main difficulty in showing

that

$$\sum T^{n'} = \infty$$

is in showing that for some k 's we have $N(k)$ such that

$$T^{k'}(w) \approx \prod_{n=0}^{N(k)} \frac{P_{n-k}(w_n)}{P_n(w_n)}$$

on a set of positive measure. For that purpose we need the following lemma which states that if k is large enough with respect to m then the derivatives of the shift under the measure $P^{(k)}$ are bounded from below up to time m on a set of large measure.

Lemma 4.2. *Given m and t there exists a $k \in \mathbb{N}$ such that*

$$P^{(k)} \left(\inf_{l \leq m} T_{(k)}^{l'}(w) \geq e^{-2^{-t}} \right) \geq 1 - 2^{-t}.$$

Proof. It follows from 2 and the structure of $P^{(k)}$ that for $l < 2^{k-1}$,

$$\begin{aligned} \log \left(T_{(k)}^{l'}(w) \right) &= \log \left(\prod_{n=0}^{2^k+l} \frac{P_{n-l}^{(k)}(w_n)}{P_n^{(k)}(w_n)} \right) \\ &= \log \left(\prod_{n=0}^{2^k+l} \left(\frac{\lambda_{n-l}^{(k)}}{\lambda_n^{(k)}} \right)^{w_n} \right) \\ &= \sum_{n=0}^{2^k+l} w_n \left(\log \lambda_{n-l}^{(k)} - \log \lambda_n^{(k)} \right). \end{aligned}$$

Using the fact that for every $n < 2^{k-1}$,

$$\lambda_n^{(k)} = \lambda_{2^k-n}^{(k)}$$

and a rearrangement of the sum one has

$$(4.2) \quad \log \left(T_{(k)}^{l'}(w) \right) = \sum_{n=0}^{2^{k-1}-l} Y_{n,k,l} + f(k,l)(w),$$

where

$$Y_{n,k,l} := \left(\log \lambda_{n-l}^{(k)} - \log \lambda_n^{(k)} \right) (w_{n+l} - w_{2^k-n})$$

and

$$f(k,l)(w) = \left(\sum_{n=2^{k-1}}^{2^k+l} + \sum_{n=0}^l + \sum_{n=2^k}^{2^k+l} \right) \left[w_n \left(\log \lambda_{n-l}^{(k)} - \log \lambda_n^{(k)} \right) \right].$$

By a trivial bound

$$(4.3) \quad |f(k, l)(w)| \leq 3l \max_{n \in \mathbb{N}} \left(\log \lambda_{n-l}^{(k)} - \log \lambda_n^{(k)} \right) \leq \frac{3l^2}{2^k}.$$

To bound the first term notice that

$$\mathbb{E}_{P^{(k)}}(Y_{n,k,l}) \propto \frac{l^2}{2^{2k}} \text{ and } \text{Var}_{P^{(k)}}(Y_{n,k,l}) \propto \frac{l}{2^{3k}}.$$

By independence of the $Y_{n,k,l}$'s we have

$$\text{Var} \left(\sum_{n=0}^{2^{k-1}-l} Y_{n,k,l} \right) \propto \frac{l^2}{2^{2k}} \ll \left(\frac{l^2}{2^k} \right) \propto \mathbb{E} \left(\sum_{n=0}^{2^{k-1}-l} Y_{n,k,l} \right).$$

It follows from this equation, Equations (4.3), (4.2) and Chebichev's inequality that if k is large enough relative to m and t then for every $l < m$,

$$P^{(k)} \left(T_{(k)}^{l'}(w) \leq e^{2^{-t}} \right) \leq \frac{e^{-t}}{m}.$$

The Lemma follows. □

Now we are ready to construct the product measure.

Let $P = \prod P_k$ where for $k \leq 0$,

$$P_k(0) = P_k(1) = \frac{1}{2}.$$

To define P_k for positive k we choose inductively two subsequences $\{n_t\}_{t \in \mathbb{N}}, \{m_t\}_{t=0}^{\infty}$ with

$$0 < n_t < m_k < n_{t+1}$$

and $m_0 = 0$. The factor measures will be fair coins for $j \in [n_t, m_t]$ and on the other segments we will choose them according to $P^{(k_t)}$.

Definition of n_t given m_{t-1} and $P|_{[m_{t-1}, n_t]}$:

By Lemma 4.2 there exists k_t such that

$$P^{(k_t)} \left(\inf_{l \leq m_t} T_{(k_t)}^{l'}(w) \geq e^{-2^{-t}} \right) \geq 1 - 2^{-t}.$$

Let $n_t = m_t + 2^{k_t}$. Now for $m_{t-1} \leq j \leq n_t$ set

$$P_j = P_{j-m_{t-1}}^{(k_t)}.$$

Definition of m_t given n_t and $P|_{[n_t, m_t]}$: Let

$$(4.4) \quad m_t = n_t + 2^{n_t}.$$

For conclusion

$$P_j(1) = 1 - P_j(0) = \begin{cases} \frac{1}{2}, & j < 0 \\ P_{j-m_{t-1}}^{(k_t)}, & m_{t-1} \leq j < n_t \\ \frac{1}{2}, & n_t \leq j < m_t \end{cases}$$

The measure satisfies

$$\liminf_{k \rightarrow \infty} P_k(1) = \frac{1}{2}$$

and

$$\begin{aligned} \limsup_{k \rightarrow \infty} P_k(1) &= \lim_{k \rightarrow \infty} P_{m_{t-1}+2^{k_t-1}}(1) \\ &= \lim_{k \rightarrow \infty} P^{(k_t)}_{2^{k_t-1}}(1) = \frac{3}{4}. \end{aligned}$$

Proposition 4.3. *The shift (X, \mathcal{B}, P, T) is conservative and ergodic and type III_1 .*

Sketch of proof: The first step will be to show that if $m < m_t$ then

$$T^{m'}(w) \geq \left(\frac{3}{2}\right)^{-n_t} \prod_{u=t}^{\infty} T^{m'}_{(k_t)}(w(t))$$

where $\{w(t)\}_{t=1}^{\infty} \subset X$ are random sequences which are independent of one another and for each t , $w(t)$ is distributed as $P^{(k_t)}$.

Then we will use Lemma 4.2 to bound $T^{m'}$ for $m \in [n_t, m_t)$ on a set of positive measure. This will give us that $\mathfrak{C} \neq \emptyset$ which by a result of Grewe, see Lemma 5.1, yields $X = \mathfrak{C}$.

Lemma 4.4. *For every $n_t \leq n < m_t$,*

$$\frac{dP \circ T^n}{dP} = T^{n'}(w) = \left(\prod_{k=1}^t \prod_{u=m_{k-1}}^{n_k-1} \frac{1}{2P_u(w_u)} \right) \cdot \left(\prod_{l=t+1}^{\infty} \prod_{u=m_{l-1}}^{n_l+n-1} \frac{P_{u-n}(w_u)}{P(w_u)} \right).$$

Proof. This is a combination of the Theorem 3.1.2 and the fact that for every $k \notin \cup_{k=1}^{\infty} [m_{t-1}, n_t)$,

$$P_k(w_k) \equiv \frac{1}{2} \quad \forall w_k \in \{0, 1\}.$$

Note that we also used the fact that for every $l > t$, and $n < m_t$

$$m_{l-1} - n > m_{l-1} - n_{l-1} > n_{l-1}$$

so the segments $[m_{l-1}, n_{l-1})$ do not overlap when we shift by n . □

Proof of Proposition 4.3. Set

$$A_t = \left\{ w \in X : \forall k \leq m_{t-1}. \prod_{u=m_t}^{n_{t+1}+k} \frac{P_{u-k}(w_u)}{P_u(w_u)} \geq e^{-2^{-t}} \right\}.$$

We have that A_1, A_2, \dots are independent and since

$$\prod_{u=m_t}^{n_t+n} \frac{P_{u-k}(w_u)}{P_u(w_u)} = \prod_{u=m_t}^{n_{t+1}+n} \frac{P_{u-m_t-n}^{(k_t)}(w_u)}{P_{u-m_t}^{(k_t)}(w_u)} = T_{(k_{t+1})}^{n'}(w|_{[m_t, n_{t+1}+n)})$$

and $P|_{[m_t, n_{t+1}+n)} = P^{(k_t)}|_{[0, 2^{k_t}+n]}$ we have by Lemma 4.2 and the choice of k_t that

$$P(A_t) = P^{(k_{t+1})} \left(\inf_{k \leq m_t} T_{(k_{t+1})}^{n'} \geq e^{-2^{-t}} \right) \geq 1 - e^{-t}.$$

Set $A = \cap_t A_t$. Then

$$P(A) \geq \prod_{t=1}^{\infty} (1 - e^{-t}) > 0.$$

For every $m_{t-1} \leq n \leq m_t$, $l > t$ and $w \in A$ we have

$$\prod_{u=m_l}^{n_{l+1}+k} \frac{P_{u-k}(w_u)}{P_u(w_u)} \geq e^{-2^{-l}}.$$

Applying the last inequality together with Lemma 4.4 we see that for $w \in A$ and $n_{t-1} \leq n \leq m_t$,

$$\begin{aligned} T^{n'}(w) &\geq \left(\prod_{k=1}^{t-1} \prod_{u=m_{k-1}}^{n_k-1} \frac{1}{2P_u(w_u)} \right) \cdot \prod_{j=l}^{\infty} e^{-l} \\ &\geq e^{-1} \prod_{k=1}^{n_t} \frac{1}{2 \cdot \frac{3}{2}} = \frac{1}{e} \cdot \left(\frac{2}{3} \right)^{n_t}. \end{aligned}$$

Therefore for every $w \in A$,

$$\begin{aligned} \sum_{n=1}^{\infty} T^{n'}(w) &\geq \sum_{t=1}^{\infty} \sum_{u=n_{t-1}}^{m_t} T^{n'}(w) \\ &\geq e^{-1} \sum_{t=1}^{\infty} \left[\left(\frac{2}{3} \right)^{n_{t-1}} (m_t - n_{t-1}) \right] = \infty. \end{aligned}$$

Here the last assertion follows from (4.4). Thus $A \subset \mathfrak{C}$. By Lemma (5.1) the shift is conservative. □

5. APENDIX

Here we give a proof of a result from [Grewe].

Lemma 5.1. *[Grewe] Let P be a product measure on X . Then if the factor measures are bounded away from 0 and 1 (e.g. $\exists p > 0$ s.t. $\forall k \in \mathbb{Z}, p < P_k(0) < 1 - p$) then the shift (X, P, T) is either conservative or dissipative.*

Proof. The condition on the factor measures ensures that for every $k \in \mathbb{Z}, w_1, x_1 \in \{0, 1\}$

$$c := \min \left(\frac{p}{1-p}, \frac{1-p}{p} \right) \leq \frac{P_k(x_1)}{P_k(w_1)} \leq c^{-1}.$$

This means that if $x, w \in \{0, 1\}^{\mathbb{Z}}$ differ in only finitely many coordinates then there exists $M > 0$ s.t

$$\frac{1}{M} T^{n'}(x) \leq T^{n'}(w) = \prod_{k=1}^{\infty} \frac{P_{k-n}(w_k)}{P_k(w_k)} \leq M T^{n'}(x).$$

Therefore

$$\sum_{n=1}^{\infty} T^{n'}(w) = \infty \Leftrightarrow \sum_{n=1}^{\infty} T^{n'}(x) = \infty$$

and so the conservative and the dissipative parts are in

$$\cap \mathcal{F}_n$$

where \mathcal{F}_n is the sub sigma algebra generated by $\{w_k : |k| \geq n\}$. By the Zero One Law $\mathfrak{C} = X$ or $\mathfrak{D} = X$. \square

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